On Superstring and Supergravity Dimensions

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Abstract. In this paper we shall show how to construct two new superstring theories in dimensions 12 and 16 which satisfy the Lorenz symmetries in 4 dimensions after compactification but which satisfy different symmetry groups before compactification. The arguments are also applied to supergravity theory. We write the first few terms of these new supergravity actions. The new supergravity actions can be expressed in terms of 3-branes and 5-branes respectively.

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1. Introduction

One of the most profound results to come from the study of superstring theory is that the number of dimensions is fixed by the theory. The theory is only consistent in 10 dimensions. We shall argue in this paper that an assumption made in constructing the superstring theory which arrives at this result can be altered and yet we still end up with a theory which is consistent and is Lorenz invariant in our 4 space-time dimensions after compactification.

2. Superstring Dimensions

To derive the number of dimensions in standard superstring theory we look for a theory in D dimensions which is Lorenz invariant in D dimensions. In other words we specify that the rotation group of the space must be O(D-1,1). We find that these constraints can only be satisfied in the quantisized theory when D=26 for bosonic string theory and D=10 for superstring theory. Next to make contact with the real world we specify that all but 4 of these dimensions are compactified. So we end up with a theory with 4 infinite dimensions which are Lorenz invariant in 4 dimensional space with rotation group O(3,1) and has a gauge group corresponding to how the other dimensions were compactified.

3. The Physics

The essential physics of this procedure is that the final theory must be Lorenz invariant in 4 dimensions to be a realistic physical theory. However the assumption that the uncompacted theory in D dimensions must also be Lorenz invariant is an assumption which need not necessarily hold true. That is, the rotation group of the D dimensional space need not necessarily be O(D-1,1). We have no physical evidence that it should be. The reason this assumption is made is by analogy to the real 4 dimensional space. However so long as there is a way to compactify the dimensions to leave 4 dimensions having rotation group O(3,1) then we are free to consider other groups. We should consider any group for which the group O(3,1) is a subgroup. Not all groups will work since only certain groups are compatible with the quantisized string theory. We can summarise this circular relation as:

$$\text{group family} \rightarrow \text{constraints} \rightarrow \text{number of dimensions} \rightarrow \text{unique group}$$

4. The Superstring groups

The main lie group families that we shall consider are the three that correspond to Lorenz transformations in real, complex and quaternion space: O(n), SU(n) and HU(n). See the appendix for alternative names for the group HU(n). These groups apply in dimensions of D=n, D=2n and D=4n respectively. After compactification they must leave the rotation group O(3,1) so this must be a subgroup.

For example if we consider the rotation group of the D=2n dimensional space to be SU(n) then this alters the constraints on the quantisized string theory. Again these constraints can only be satisfied in a certain number dimensions. We shall derive the number of dimensions in several different ways.

5. Light Cone Gauge

Firstly we use the straight forward intuitive method of using the light-cone gauge. In the light cone gauge, to avoid ghosts, we set:

$$X^{\pm} = X^0 \pm X^1$$

which leaves D-2 transverse dimensions over which we sum when we calculate the string modes. In real space the anomalous factor to be considered is:

$$1 - \frac{D-2}{24}$$

which becomes zero if we set D=26. In complex string theory we set the complex light-cone co-ordinates to:

$$Z^{\pm} = Z^0 \pm Z^1$$

leaving D-4 transverse dimensions. (We count each complex dimension counts as 2 dimensions - see appendix). The anomalous factor is then:

$$1 - \frac{D-4}{24}$$

This becomes zero when we set D=28 which makes the Lorenz group for complex bosonic string theory SU(14). For quaternion space the anomalous term is:

$$1 - \frac{D-8}{24}$$

which gives D=32 and Lorenz group HU(8). Since there are no more lie group families we can't go any further than this.

6. Superstrings

For superstring theory the anomalous parts that must be set to zero are

$$1 - \frac{D-2}{8}, 1 - \frac{D-4}{8}, 1 - \frac{D-8}{8}$$

respectively, which gives the groups O(10), SU(6) and HU(4) in dimensions D=10, D=12 and D=16. We have outlined the argument in a simple intuitive way in order to show how to simply calculate the number of dimensions of the three string theories.

7. Complex Dimensions

So we find that for bosonic strings we have a maximum of 32 uncompacted dimensions, and for superstrings we have a maximum of 16 uncompacted dimensions. It is hard for us to imagine a space of higher than 4 dimensions let alone one with a complex Lorenz group structure but there are no physical reasons for dismissing such spaces so long as the compactified model restores the 4 dimensional space with Lorenz group O(3,1).

8. Compactificaction

When we compactify all but 4 dimensions we need to be careful which dimensions we choose. For example for the case of SU(6) if we compactify all but 2 complex dimensions that would leave the group SU(2)=O(3) which is smaller than O(3,1). We can get around this by compacting all but one component of 4 complex dimensions giving an O(4) group. In the case of HU(4) we can compactify all complex parts of the dimensions to leave an O(4) group.

9. Three Models

We have shown the existence of three superstring theories with uncompacted rotation groups of O(10), SU(6) and HU(4) so which one, if any, represents reality and what about the other two? If the progress of science is anything to go by then theories with higher symmetries tend to be less prone to inconsistencies. Therefor we would expect HU(4) superstring theory to be the closest to reality since it is the only one with the additional quaternion symmetry. If this is the case then we would expect that the other two superstring theories contain infinities. Of course the fact that its dimensionality is 4 suggests that we may find a simple explanation of why all but 4 of these dimensions are compactified.

10. Physics in Complex Space

What exactly is physics in complex space? For the most part physics in D complex dimensions is the same as physics in 2D real dimensions. In complex space, however, the physics of an object is not necessarily the same for every O(2D) rotation we make on it. Only a subset of these rotations namely the ones that form the group SU(D) leave its physics unchanged. Subsequently there are more distinct geometric objects in complex space because they can't be identified with each other by an O(2D) rotation. Fortunately due to the identity O(2)=U(1), the physics of 2-dimensional space is the same in both cases.

As a more complicated example, imagine embedding a 4 dimensional object such as a string world path into 2 complex dimensions. Because of the identity O(4)=SU(2)xSU(2) this means that depending on the orientation of the string in real space you actually create a family of string paths in complex space, none of which can be identified via complex rotation. We can label these objects with the 3 parameters of the SU(2) group which is not being used. Hence our intuitive picture of a string world path in 4 dimensions becomes a family of distinctly different world paths in 2 complex dimensions each with its own unique probability amplitude which is a function of the 3 left over parameters.

Some readers may object to the notion of complex time, however this isn't a problem so long as we realise that the theory only makes sense when the extraneous dimensions are compacted and we are left with 3+1 real space-time dimensions. After that we can treat the time variable as usual.

11. Complex Superstring theory

String theory in complex space is constructed with analogy to string theory in real space but now unfortunately we no longer have the direct analogy to vibrating strings that we find in our 3+1 dimensional real space-time. We have to use mathematical consistency alone. The Polyakov action [3] for complex bosonic string theory is:

$$S = -\frac{1}{4\pi\alpha'}\int_M (-\gamma)^{-1/2}\gamma^{ab}\partial_a Z_\mu \partial_b \overline{Z}^\mu d\sigma d\tau$$

where the space-time co-ordinates are now complex. Most of string theory can easily be carried over into complex space. The main differences being the introduction of conjugates and a different number of dimensions. In fact, the spectrums of the three string theories have the same number of particles (since this is down to the conformal properties of the string and not how many space-time dimensions the string exists in). For example the Type I strings all have 128 bosons and 128 fermions at the first level suggesting that they correspond to a supergravity theory. The interesting bit comes when we compactify extraneous dimensions to get the gauge groups in 4 dimensional space-time.

12. Heterotic Strings

These types of string theory can be thought of as combining the bosonic string for clockwise moving strings and the superstring for anticlockwise moving strings. The difference between the number of dimensions of the two theories must be compactified on a lattice. For the 10 dimensional heterotic string we must compactify 26-10=16 dimensions. We end up with a gauge group of O(32) or E8xE8 depending on which even self-dual lattice we chosen. Even self-dual lattices exist only in 8n dimensions. For all our new string theories the number of dimensions to be compactified for their heterotic versions also turns out always to be 16.

13. Supergravity

Supergravity can be thought of either as superstring theory in some limit or as the low energy limit of a new 11 dimensional theory of membranes referred to as M-Theory. Either way supergravity is always one (complex) dimension larger than the equivalent string theory. Thus we expect to find two new supergravity theories in complex and quaternion space: SU(7/1) supergravity and HU(5/1) supergravity. As in string theory all but 4 of these dimensions must be compacted to leave a 4 dimensional real space with O(3,1) Lorenz symmetry.

To show the plausibility of this statement we note that 11 dimensional supergravity is constructed from three objects:

$$\{e_M^A, A_{MNP}, \psi_M\}$$

Supersymmetry requires that the number of independent boson components is equal to the number of independent fermion components. We can use this result to construct supergravity theories with complex and quaternion Lorenz symmetries. In complex space, for example, g^{mn} becomes a hermitian tensor. These new restrictions alter the number of independent components of the fields and also alters the number of dimensions that the boson components become equal to the fermion components.

The number of fermion components for real supergravity is given by:

$$(d-3)2^{[d/2]-1}$$

which are the number of gauge-invariant components of a Majorna-Weyl spin 3/2 field in d dimensions. For d=11 this gives 128 components. When we extend this into complex space the dimensions we are interested in occur when d-3 is a power of 2. For d-3=8, 4 and 2 we expect to find supergravity theories OSp(11/1), SU(7/1) and HU(5/1). These are all 1 (complex) dimension larger than their equivalent superstring theories which is due to the quantity d-3 being considered as opposed to d-2 for string theory. These three supergravity theories are the largest possible in their respective categories that have fields with spins no higher than 2 when compacted down to 4 dimensions. This is shown by noting that there are 8 steps between spins of -2 and +2. We calculate the maximum number of steps for the respective theories as follows: (11-3)=8 for real supergravity, 2x(7-3)=8 for complex supergravity and 4x(5-3)=8 for quaternion supergravity. [2] All three theories have precisely 128 bosons and 128 fermions. We might be tempted to say that therefor they are the same theory, however the particles fall into different groupings for each theory and the resulting gauge groups that we get from compactifying the dimensions down to 4 is different for each theory. SU(7/1) and HU(5/1) supergravity theories also have additional symmetries not present in OSp(11/1) supergravity.

After compactifiaction to 4 dimensions we are left with maxmimum gauge groups of O(8), SU(6) and HU(5) respectively. It is interesting to note that the last two groups SU(6) and HU(5) contain within them the minimal guage group of the Standard Model U(1)xSU(2)xSU(3) which is a distinct advantage over real supergravity. Of course we get the down side of having to deal with complex space, however this can be sensibily dealt with if the correct compactification is found so as to leave 3+1 dimensional real space.

To find the action for each of these supergravity theories we can examine the low energy limits for the equivalent string theory and then extrapolate from there. As in the case of OSp(11/1) supergravity we don't expect these to be renormalizable theories. However they serve as a useful basis in the investigation of string theory and M-Theory. We will write the first few terms of these actions:

14. OSp(11/1) Supergravity Action

We will briefly review the 11 dimensional supergravity action. This is made from three objects:

$$\{e_M^A, A_{MNP}, \psi_M\}$$

The presence of the 3rd rank tensor tells us that this theory contains membranes (2-branes). It is thought to be the low energy limit of a new 11 dimensional membrane theory called *M-Theory*. The action for this begins [2]:

$$S_{Osp(11/1)} = \int dx^{11}(\frac{1}{\kappa^2}eR + a_1 eF^2 + a_2\kappa\varepsilon^{M_1..M_{11}}F_{M_1..M_4}F_{M_5..M_8}A_{M_9..M_{11}} + O(\psi))$$

where F is the curl of A. The a's are certain defined constants.

15. SU(7/1) Supergravity Action

We can derive this either by considering the link to SU(6) string theory or from first principles by brute force by counting components. We find that it includes these three objects:

$$\{e_M^A, A_{PQ}^{MN}, \psi_M\}$$

The *vierbein*, e, is now a hermitian tensor and A is antisymmetric in two pairs of indicies and anti-hermitian in the other two. The presence of the 4th rank tensor also tells us that this theory contains 3-branes in 14 dimensions. The action for this begins:

$$S_{SU(7/1)} = \int dz^7 d\overline{z}^7(\frac{1}{\kappa^2}eR + b_1 e|F|^2 + b_2\kappa\varepsilon^{N_1..N_7}\varepsilon_{M_1..M_7}F_{M_1M_2}^{N_1N_2N_3}\overline{F}_{M_3M_4M_5}^{N_4N_5}A_{M_6M_7}^{N_6N_7} + ...)$$

We can think of it as the low energy limit of a 14 dimensional theory of 3-branes which we shall call *C-Theory* (where C stands for complex).

16. HU(5/1) Supergravity Action

We find that it includes these three objects:

$$\{e_M^A, A_{QRS}^{MNP}, \psi_M\}$$

The tensors now have quaternion-hermitian and quaternion-antihermitian symmetries. The presence of the 6th rank tensor tells us that this theory contains 5-branes in 20 dimensions. The action for this begins:

$$S = \int |dq|^{20}(\frac{1}{\kappa^2}eR + b_1 e|F|^2 + b_2\kappa f^{N_1..N_{10}}_{M_1..M_{10}} F^{N_1N_2N_3N_4}_{M_1M_2M_3} \overline{F}^{N_5N_6N_7}_{M_4M_5M_6M_7} A^{N_8N_9N_{10}}_{M_8M_9M_{10}} + ...)$$

Where f is a constant tensor which preserves the quaternion symmetry. We can think of it as the low energy limit of a 20 dimensional theory of 5-branes which we shall call *Q-Theory* (where Q stands for quaternion). Because of the non-commutative nature of quaternions, this action is the most difficult to write and calculate. Mixing non-commutative space-dimensions with anti-commutative super-space dimensions creates some interesting and nonintuitive objects such as *quaternion Grassman numbers* which have intriguing properties such as $\theta^4 = \theta^2\overline{\theta}^2 = 0$ and $\theta^3\overline{\theta} \neq 0$. Hence although theoretically this theory is the most symmetric of all three supergravity actions, what we gain in symmetry we lose in the intuitiveness of the action.

17. Compactification

Each supergravity action is one complex dimension larger than the equivalent string theory. Thus if we compactify one complex dimension of each theory we should get a string theory. For M-Theory compactifying one real dimension turns the membranes (2-branes) into strings (1-branes). We get 10 dimensional superstring theory.

For C-Theory we compactify one complex dimension which is the same as two real dimensions, so this turns the 3-branes into 1-branes so this gives us SU(6) superstring theory.

For Q-Theory we compactify one quaternion dimension which is the same as 4 real dimensions so this turns the 5-branes into 1-branes so this gives us HU(4) superstring theory.

18. Dual Branes

Each theory consists of p-branes and their duals (q-branes). These arise from the dual of the F tensor and so the simple formula to connect p-branes and their duals is:

$$p + q + 4 = D$$

From this we deduce that M-Theory includes Membranes (2-branes) and 5-branes (as well as other objects) where D=11.

C-Theory includes 3-branes and 7-branes (as well as other objects) where D=14.

Q-Theory includes 5-branes and 11-branes (as well as other objects) where D=20;

19. Conclusion

We have shown that an assumption in constructing superstring theory which leads to the result that it is only consistent in 10 uncompacted dimensions can be changed and this change leads to other superstring theories with 12 or 16 uncompacted dimensions. These superstring theories possess higher symmetries. We also get different gauge groups when we compactify each theory down to 4 dimensions. We summarise the results below in this table. These are the primary results of this paper.

20. Table 1

Bosonic String	**Supergravity**	**Superstring**
O(26),D=26	OSp(11/1),D=11	O(10),D=10
SU(14),D=28	SU(7/1),D=14	SU(6),D=12
HU(8),D=32	HU(5/1),D=20	HU(4),D=16

Appendix A. Appendix

The real, complex, quaternion number fields are represented by **R**, **C**, **H** where H stands for 'hypercomplex'. The quaternion Lorenz group HU(n) is known by other names as U(n,**H**), Sp(n), Sp(2n), Usp(2n). We have used this notation in order to emphasise it as a group of quaternion dimension n. HU(n) is the group of quaterion-unitary matrices applied on the left only. To include all quaternion rotations we can include unit quaternions multiplied on the right. This extends the group to HU(n)x**H**.

The factor 1/24 comes in light-cone string theory comes from summing the infinite number of modes of the string using zeta function regularisation:

$$\sum_{n=1}^{\infty} \frac{n}{2} = \frac{1}{2}\zeta(-1) = -\frac{1}{24}$$

The term 'dimension' in this paper refers to any independent degree of freedom. For example we can talk of a complex number as being 2 dimensional and a quaternion as being 4 dimensional. When we talk of 'complex dimensions' however we are referring to pairs of variables. For example a complex number has 1 complex dimension. The variable D always refers to 'dimensions', a small d refers to complex dimensions.

References

[1] Polchinski, Joseph, *String Theory (Volume 1),* (Cambridge University Press, 1998).
[2] Kaku, Michio *Introduction to Superstrings and M-Theory,* (Springer Verlag New York, 1999).
[3] Polyakov, A.M, Phys Lett. **103B**, 207 211 (1981).
[4] Sethi, Savdeep, *A Survey of High-Dimensional Supergravity,* (July 2001).